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# Relativistic Coulomb Resummation in QCD

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A relativistic Coulomb-like resummation factor in QCD is suggested, based on the solution of the quasipotential equation.

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In describing a charged particle-antiparticle system near threshold, it is well known from QED that the so-called Coulomb resummation factor plays an important role [1]. This resummation, performed on the basis of the nonrelativistic Schrödinger equation with the Coulomb potential $V(r) = -\alpha/r$, leads to the Sommerfeld-Sakharov factor [2,3]

$$S_{\rm nr} \,=\, \frac{X_{\rm nr}}{1\,-\,\exp(-X_{\rm nr})}\,, \qquad X_{\rm nr} \,=\, \frac{\pi\,\alpha}{v_{\rm nr}}\,, \tag{1}$$

which is related to the wave function of the continuous spectrum at the origin, $|\psi(0)|^2$. Here $v_{\rm nr}$ is the velocity of the particle. An expansion of Eq. (1) in a power series in the coupling constant $\alpha$ reproduces the threshold singularities of the Feynman diagrams in the form $(\alpha/v)^n$. However, in the threshold region one cannot truncate the perturbative series and the $S$-factor should be taken into account in its entirety. A description of quark-antiquark systems near threshold, which has now been intensively investigated [4], also requires this Coulomb resummation. The $S$-factor appears in the parametrization of the imaginary part of the quark current correlator, the Drell ratio $R(s)$, which can be approximated in terms of the Bethe-Salpeter (BS) amplitude of two charged particles $\chi_{\rm BS}(x)$ at $x=0$ [5]. The nonrelativistic replacement of this amplitude by the wave function which obeys the Schrödinger equation with the Coulomb potential, leads to the approximation (1) with $\alpha \to 4\alpha_s/3$, for QCD.

In the relativistic theory, especially for systems composed of quarks lighter than the top, the nonrelativistic approximation needs to be modified. To use the $S$-factor within such a relativistic regime one usually uses the simple substitution $v_{\rm nr} \to v$ with $v = \sqrt{1-4m^2/s}$. However, the corresponding relativistic generalization of the $S$-factor is obviously not unique, for there are numerous ways of expressing the nonrelativistic velocity in terms of the relativistic energy $\sqrt{s}$. For a systematic relativistic analysis of quark-antiquark systems, it is essential from the very beginning to have a relativistic generalization of the $S$-factor.

In this letter we suggest a new form for this relativistic factor in the case of QCD. Our starting point is the quasipotential (QP) approach proposed by Logunov and Tavkhelidze [6], in the form suggested by Kadyshevsky [7]. To find an explicit form for the relativistic $S$-factor we will transform the QP equation from momentum space into relativistic configuration space [8]. The local Coulomb potential defined in this representation has a QCD-like behavior in momentum space [9].

The possibility of using the QP approach to define the relativistic $S$-factor is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$, is taken at $x=0$, therefore, in particular, at relative time $\tau=0$. The QP wave function is defined as the BS amplitude at $\tau=0$, and the $R$-ratio can be expressed through the QP wave function $\psi_{\rm QP}(\mathbf{p})$ by using the relation

$$\chi_{\rm BS}(x=0) \,=\, \int d\Omega_p\, \psi_{\rm QP}(\mathbf{p})\,, \tag{2}$$

where $d\Omega_p = (d\mathbf{p})/[(2\pi)^3\,E_p]$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_p^2 - \mathbf{p}^2 = m^2$.[1]

The QP equation in momentum space has the form

$$(2E \,-\, 2E_p)\;\psi(\mathbf{p}) \,=\, \int d\Omega_k\, V\,(\mathbf{p}(-)\mathbf{k})\;\psi(\mathbf{k})\,. \tag{3}$$

The proper Lorentz transformation, $\Lambda_{\mathbf{k}}$, means a translation in the Lobachevsky space

$$\Lambda_{\mathbf{k}}\,\mathbf{p} \,\equiv\, \mathbf{p}(+)\mathbf{k} \,=\, \mathbf{p} + \mathbf{k}\left[\sqrt{1+\mathbf{p}^2} \,+\, \frac{\mathbf{p}\cdot\mathbf{k}}{1+\sqrt{1+\mathbf{k}^2}}\right]. \tag{4}$$

The role of the plane waves corresponding to the translations (4) is played by the following functions

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[1] In the following we will consider the case of two scalar particles with the same masses $m$ and use the system of units $c = \hbar = m = 1$.

$$\xi(\mathbf{p},\mathbf{r}) \,=\, (E_p - \mathbf{p}\cdot\mathbf{n})^{-1-ir}\,, \tag{5}$$

where $\mathbf{r} = \mathbf{n}r$ and $\mathbf{n}^2 = 1$. These functions correspond to the principal series of unitary representations of the Lorentz group and in the nonrelativistic limit ($p \ll 1$, $r \gg 1$) $\xi(\mathbf{p},\mathbf{r}) \to \exp(i\mathbf{p}\cdot\mathbf{r})$. The orthogonality and completeness relations for these functions are

$$\begin{aligned}&\int d\Omega_p\, \xi(\mathbf{p},\mathbf{r})\,\xi^*(\mathbf{p},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\,,\\ &\int (d\mathbf{r})\, \xi(\mathbf{p},\mathbf{r})\,\xi^*(\mathbf{k},\mathbf{r}) = (2\pi)^3\delta(\mathbf{p}(-)\mathbf{k})\,,\end{aligned} \tag{6}$$

where the relativistic momentum-space $\delta$-function is $\delta(\mathbf{p}(-)\mathbf{k}) = \sqrt{1+\mathbf{p}^2}\,\delta(\mathbf{p}-\mathbf{k})$. The QP wave functions in the momentum and relativistic configuration representations are related as follows:

$$\begin{aligned}\psi(\mathbf{r}) &= \int d\Omega_p\, \xi(\mathbf{p},\mathbf{r})\,\psi(\mathbf{p})\,,\\ \psi(\mathbf{p}) &= \int (d\mathbf{r})\, \xi^*(\mathbf{p},\mathbf{r})\,\psi(\mathbf{r})\,.\end{aligned} \tag{7}$$

For a spherically symmetric potential the $\xi$-transform of Eq. (3) is the equation

$$\begin{aligned}&\int d\Omega_p\,(d\mathbf{r}')\,(2E-2E_p)\,\xi(\mathbf{p},\mathbf{r})\xi^*(\mathbf{p},\mathbf{r}')\,\psi(\mathbf{r}')\\ &\quad = V(r)\,\psi(\mathbf{r}),\end{aligned} \tag{8}$$

where the right hand side is local. Here the transform of the potential is given in terms of the same relativistic plane wave,

$$V(\mathbf{p}(-)\mathbf{k}) = \int (d\mathbf{r})\,\xi^*(\mathbf{p}(-)\mathbf{k},\mathbf{r})V(\mathbf{r}). \tag{9}$$

The left hand side of this equation can be rewritten in a non-integral form by using the operator of the free Hamiltonian [8]

$$\hat{H}_0 \,=\, \cosh\left(i\frac{d}{dr}\right) + \frac{i}{r}\sinh\left(i\frac{d}{dr}\right) - \frac{\Delta_{\theta,\varphi}}{2r^2}\exp\left(i\frac{d}{dr}\right)\,, \tag{10}$$

where $\Delta_{\theta,\varphi}$ is the angular part of the Laplacian operator. The relation $\hat{H}_0\xi(\mathbf{p},\mathbf{r}) = E_p\xi(\mathbf{p},\mathbf{r})$ allows one to re-express the equation in terms of finite differences

$$\left(2E - 2\hat{H}_0\right)\psi(\mathbf{r}) \,=\, V(r)\,\psi(\mathbf{r})\,. \tag{11}$$

This equation for the Coulomb potential has been investigated in Ref. [10]. The solutions contain arbitrary functions of $r$ with period $i$, the so-called the $i$-periodic constants, which appear in the solutions due to the finite difference nature of the Hamiltonian (10). For some problems, such as defining the bound state spectrum, this $i$-periodic constant is not important. However, for the purpose of extracting the $S$-factor, we must develop a method which avoids this ambiguity.

Consider the Coulomb potential defined in relativistic configuration space

$$V(r) \,=\, -\frac{\alpha}{r}\,. \tag{12}$$

The $\xi$-transformation of Eq. (12) gives in momentum space the following function

$$V(\Delta) \,\sim\, \frac{1}{\chi_\Delta \sinh\chi_\Delta}\,, \tag{13}$$

where the relative rapidity $\chi_\Delta$ corresponds to $\mathbf{\Delta} = \mathbf{p}(-)\mathbf{k}$ and is defined in terms of the square of the momentum transfer by $Q^2 = -(p-k)^2 = 2(\cosh\chi_\Delta - 1)$. For large $Q^2$ the potential $V(\Delta)$ behaves as $(Q^2\ln Q^2)^{-1}$, which reproduces the principal behavior of the QCD potential proportional to $\bar{\alpha}_s(Q^2)/Q^2$ with $\bar{\alpha}_s(Q^2)$ being the QCD running coupling.

According to Eqs. (2), (5), and (7), we find a relation between the required BS amplitude and the QP wave function, $\chi_{\mathrm{BS}}(x=0) = \psi_{\mathrm{QP}}(r=i)$. Performing a partial-wave analysis we further observe that the QP wave function for an $\ell$-state will contain the generalized power $(-r)^{(\ell+1)} = i^{l+1}\Gamma(ir+l+1)/\Gamma(ir)$, which vanishes at $r = i$ for $\ell \neq 0$. Thus, we need only to consider the $\ell = 0$ wave function for which we can write $\psi(\mathbf{r}) = \psi(r)$. Introducing the function $R(r) = r\,\psi(r)$ into Eq. (8), we get

$$\begin{aligned}&\frac{2}{\pi}\int_0^\infty d\chi' \int_0^\infty dr'\,\sin\chi' r\,\sin\chi' r'(2E - 2\cosh\chi')R(r')\\ &\quad = V(r)\,R(r)\,.\end{aligned} \tag{14}$$

We will seek a solution of Eq. (14) with the Coulomb potential (12) in the form

$$R(r) \,=\, \int_\alpha^\beta d\zeta\,\exp(i\,r\,\zeta)\,R(\zeta)\,, \tag{15}$$

where the $\zeta$-integration is performed in the complex plane over a contour with endpoints $(\alpha,\beta)$ [11]. Substituting Eq. (15) into Eq. (14) we find the equation[2]

$$\begin{aligned}&\int_\alpha^\beta d\zeta\,\exp(i\,r\,\zeta)\,\left(2E - 2\cosh\zeta\right)\,R(\zeta)\\ &\quad = -\frac{\alpha}{r}\int_\alpha^\beta d\zeta\,\exp(i\,r\,\zeta)\,R(\zeta)\,,\end{aligned} \tag{16}$$

which, when we integrate by parts, yields the two equations

[2] We first perform the $\chi'$-integration with a regularization factor $\exp(-\epsilon\chi'^2)$ and then set $\epsilon = 0$ after all calculations.

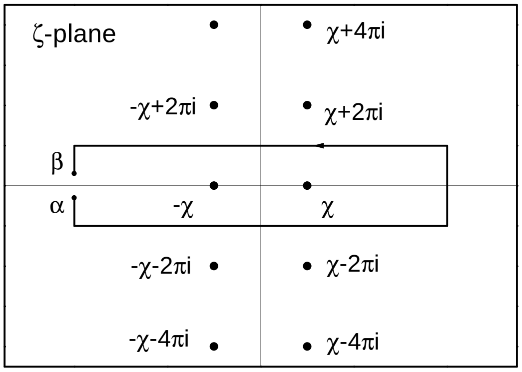


FIG. 1. *Contour of integration in Eq. (15) and singularities of the function (19) in the complex $\zeta$-plane.*

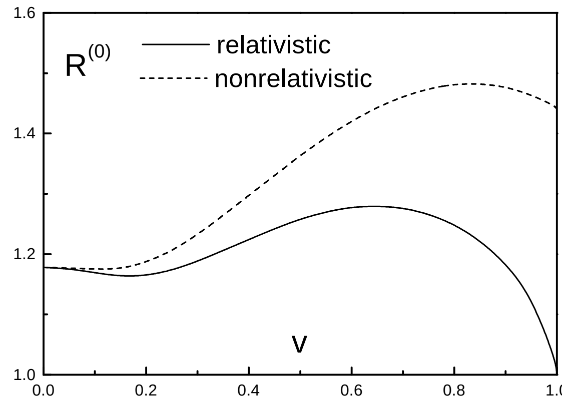


FIG. 2. *Behavior of $R^{(0)}(s)$ with relativistic and nonrelativistic $S$-factors.*

$$\exp(i\,r\,\zeta)\;(\,2E - 2\cosh\zeta)\;R(\zeta)\Big|_{\zeta=\alpha}^{\zeta=\beta} \;=\; 0 \tag{17}$$

and

$$i\,\frac{d}{d\zeta}\left[(\,2E - 2\cosh\zeta)\;R(\zeta)\right] \;=\; -\,\alpha\,R(\zeta)\,. \tag{18}$$

The solution of Eq. (18) is

$$R(\zeta) = C(\chi)\;\exp(\zeta)\Big[\exp(\zeta)\;-\;\exp(-\chi)\Big]^{A-1} \times\Big[\exp(\zeta)\;-\;\exp(\chi)\Big]^{-A-1}, \tag{19}$$

where $A = i\alpha/(2\sinh\chi)$, $E = \cosh\chi$, and $C(\chi)$ is an arbitrary function of $\chi$.

The branch points of the function (19) are $\pm\chi + 2\pi i n$ (see Fig. 1). The contour of integration must not intersect cuts which we take from $-\infty + 2\pi i n$ to $\pm\chi + 2\pi i n$. In the case when the interaction vanishes, $\alpha \to 0$, the solution $R(r)$ should reproduce the known free wave function $\sin\chi r/\sinh\chi$. Taking into account these remarks and Eq. (17) for the boundary values at $(\alpha,\beta)$, we take $\alpha = -R - i\varepsilon$, $\beta = -R + i\varepsilon$ with $R \to \infty$. The vertical part of the contour to the right is given by $\mathrm{Re}\,\zeta = +R$. It is also convenient for finding a connection to an integral representation of the hypergeometric function to take the horizontal parts of the contour to be characterized by $\mathrm{Im}\,\zeta = \pm\pi$ (see Fig. 1).

The resulting solution does not contain the $i$-periodic constant and reads[3]

$$R(r) = C(\chi)\;\sinh\pi r\int_{-\infty}^{\infty} dx\;\exp(irx)\;\exp(x) \times\Big[\exp(x)\;+\;\exp(-\chi)\Big]^{A-1}\Big[\exp(x)\;+\;\exp(\chi)\Big]^{-A-1}. \tag{20}$$

Comparing the asymptotic form of Eq. (20) at $r \to \infty$ with the free wave function we can determine the constant $C(\chi)$ and calculate $|\psi_{\rm QP}(i)|^2$ which leads to the relativistic $S$-factor:

$$S(\chi) \;=\; \frac{X(\chi)}{1\;-\;\exp\left[-X(\chi)\right]}\,, \qquad X(\chi) \;=\; \frac{\pi\,\alpha}{\sinh\chi}\,, \tag{21}$$

where $\chi$ is the rapidity which related to $s$ by $2\cosh\chi = \sqrt{s}$. The function $X(\chi)$ in Eq. (21) can be expressed in terms of $v$ as $X(\chi) = \pi\alpha\sqrt{1-v^2}/v$.

We note that this new relativistic factor could have a significant impact in interpreting strong-interaction physics. In many physically interesting cases, $R(s)$ occurs as a factor in an integrand, as, for example, for the case of inclusive $\tau$ decay, for smearing quantities, and for the Adler $D$ function. Here the behavior of $S$ at intermediate values of $v$ becomes important. To illustrate the difference between the factors (1) and (21), in Fig. 2 we plot the principal contribution to $R(s)$ for the vector currents,

$$R^{(0)}(s) = \frac{v(3-v^2)}{2}S, \tag{22}$$

for the nonrelativistic case with $v_{\rm nr} \to v$ and the relativistic one, for $\alpha = 0.25$.

We conclude this letter by discussing the two limiting cases. In the nonrelativistic limit, $v \ll 1$, the relativistic $S$-factor (21) reproduces the nonrelativistic result (1). In the ultrarelativistic limit, as it has been argued in Ref. [12], the bound state spectrum vanishes

[3] The representation of this solution in terms of the hypergeometric function can be found, for instance, by the substitution $x = \chi - \ln s$.

as $m \to 0$ because the particle mass is the only dimensional parameter. This feature reflects an essential difference between potential models and quantum field theory, where an additional dimensional parameter appears. One can conclude that within a potential model, the $S$-factor which corresponds to the continuous spectrum should go to unity in the limit $m \to 0$. Thus, the relativistic resummation factor $S$ obtained here reproduces both the expected nonrelativistic and ultrarelativistic limits and corresponds to a QCD-like Coulomb potential.

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